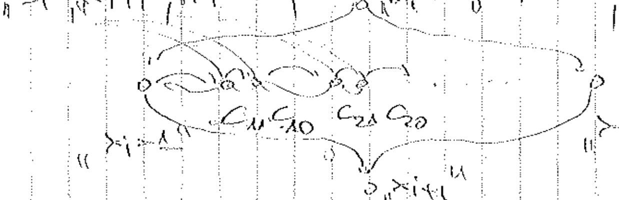
Consider the component $G_i$ for variable $x_i$. Denote nodes as "$x_i$", "$x_{i+1}$", "$x_i=1$", "$x_i$", "$x_i=0$", $c_{11} c_{10}, \dots, c_{k1}, c_{k0}$ (*)

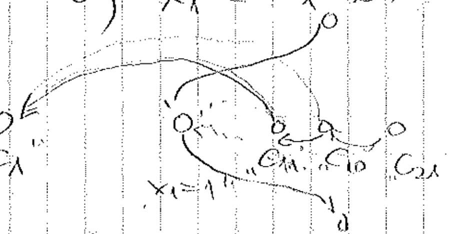



We must visit $x_i$ and go to $x_i=1$ or $x_i=0$, say $x_i=1$.

Suppose we don't go to $c_{11}$, then $x_{i+1}$ is the only choice.

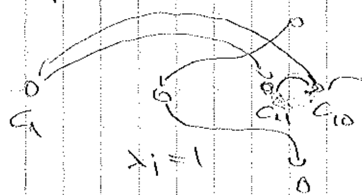
We must visit $c_1$ ...

α) $x_1 \in C_1$ => $C_{10} C_{11} C_1$ , $C_{10} C_{21}$

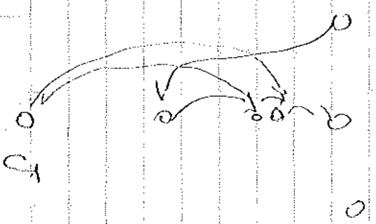



... ⟶ $x_i = 0$ ↯

β) $\bar{x}_1 \in C_1$ . $C_{10} C_{11}$ ↯ "$C_1 C_{11} C_{10} C_{21}$"

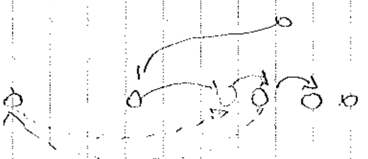



=> $x_1 x_i = 1$ $C_{11}$




α) $x_i \in C_1$ => $C_{11} C_1 C_{10} C_{21}$

$C_{11} C_{10} C_{21}$



β) $\bar{x}_i \in C_1$ => $C_{11} C_{10}$

The case $C_{11} C_{10} C_1$ leads to a contradiction as before.

=> $C_{11} C_{10} C_{21}$

Therefore $G_i$ must be traversed either in direction 1 or 0, potentially sidestepping to clause nodes.

The resulting directed Hamiltonian cycle induces a satisfying truth assignment.

(*) We show: $G_i$ must be traversed as $x_i$, $x_i=1$, $C_{11}, C_1, C_{10}, \dots, C_{k1}, C_{k0}, x_i=0, x_{i+1}$ where $C_j$ is optionally visited if $x_i \in C_j$ and left out otherwise, or in direction $x_i=0$ as $x_i, x_i=0, C_{k0}, C_{k1}, \dots, C_{20}, C_{21}, C_{10}, C_1, C_{11}, x_i=1, x_{i+1}$ where $C_j$ is optionally visited if $\bar{x}_i \in C_j$ and skipped otherwise.

9

# 2. Location Problems

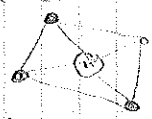
21.04.12

## 2.1 Introduction

**2.1.1 Motivation:** given a set V of points (in $\mathbb{R}^d$, in network N), find a number of medians (in $\mathbb{R}^d$, in N), i.e., new points, that minimize (sum, max) the distance ($\ell_2, \ell_2^2, \ell_1, \ell_\infty$) to V.

a) Median point in the plane (Fermat [17th century]): given a triangle, find a median (point in the plane) that min the sum of the dist. to the $\Delta$-vertices.

b) Location-allocation problem (Weber [20th cent]): As Fermat, but $n \geq 3$ points, $p \geq 1$ facilities (median), distance weights $w_p$ to account for customer demands



c) Absolute median problem (Hakimi [1960s]): V $\hat{=}$ vertices of a graph, medians $\hat{=}$ points on vertices and edges, dist $\hat{=}$ dist in graph.

**2.1.2 Def.** (Classification Scheme for Location Problems, Hamacher & Nickel [1998]):

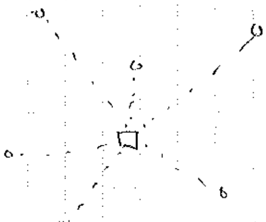
Input: new locations / domain / specifics / distances / objective

- new locations: $p = 1$ (one), $n$ (many)
- domain: $\mathbb{R}^d$, $\mathbb{R}^2$ (planar), N (network), D (discrete $\hat{=}$ finite set)
- specifics: R = ... (restricted positions), B = ... (barriers)
- distances: $\ell_2, \ell_2^2, \ell_1, \ell_\infty$
- objective: $\Sigma$ (of distances), max (of dist.)
- given points: finite $V \subseteq$ domain (implicit)

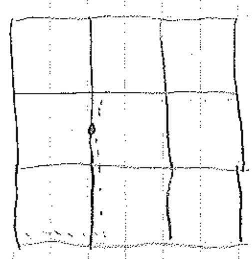
Output: $P = \operatorname{argmin}$ obj, $P \subseteq$ domain, specifics

**2.1.3 Ex.** (Schöbel & Schmidt [2008]):

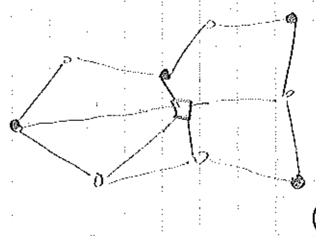
a) Desert well: $1 / \mathbb{R}^2 / \cdot / \ell_2 / \Sigma$



b) Manhattan fire brigade: $1 / \mathbb{R}^2 /$ R = buildings $/ \ell_1 /$ max

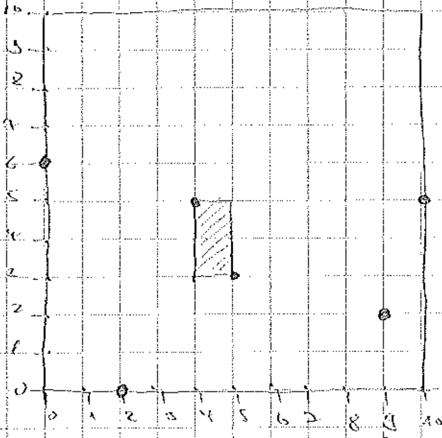


c) Warehouses: $n$ / N / $\cdot$ / shortest path / $\Sigma$



## 2.2. Medians in the Plane

**2.2.1 Ex.** (Manhattan distances, Schöbel & Schmidt[2009]):

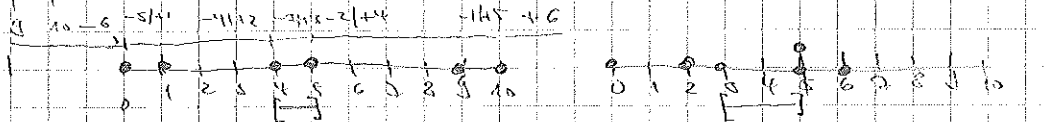


$V=\{(a_i,b_i)\}_{i=1}^{6} = \{(0,6),(4,5),(10,5),(5,3),(9,2),(2,0)\}$

$\underbrace{1}/\mathbb{R}^2/\cdot/l_1/\Sigma$

$(x,y)$

$$\min \sum_{i=1}^{6}\left(|x-a_i| + |y-b_i|\right)$$
$$= \min \sum_{i=1}^{6} |x-a_i| + \min \sum_{i=1}^{6} |y-b_i| \quad \text{(separable)}$$



$$= [\operatorname{argmin}_3 \{a_i\}, \operatorname{argmin}_4 \{a_i\}] \times [\operatorname{argmin}_3 \{b_i\}, \operatorname{argmin}_4 \{b_i\}]$$
$$= \operatorname{med}\{a_i\} \times \operatorname{med}\{b_i\}.$$

**2.2.2 Def.** (Median of numbers): Let $\{x_1,\dots,x_m\} \subseteq \mathbb{R}$ be a set of $m \in \mathbb{N}$ numbers. Then
$$\underset{\text{def!}}{\operatorname{med}} \{x_i\}_{i=1}^m := \begin{cases} \{\operatorname{argmin}_{\lceil \frac{m}{2} \rceil} \{x_i\}\}, & m \text{ odd} \\ [\operatorname{argmin}_{\frac{m}{2}} \{x_i\}, \operatorname{argmin}_{\frac{m}{2}+1} \{x_i\}], & m \text{ even} \end{cases}$$

**2.2.3 Thm** (explicit solution formula for $1/\mathbb{R}^2/\cdot/l_1/\Sigma$): Let $V=\{(a_i,b_i)\}_{i=1}^m$

$$\operatorname*{argmin}_{(x,y)\in\mathbb{R}^2} 1/\mathbb{R}^2/\cdot/l_1/\Sigma = \operatorname{med}\{a_i\}_{i=1}^m \times \operatorname{med}\{b_i\}_{i=1}^m. \quad \square$$

**2.2.4 Ex** ($l_2^2$): $V=\{(a_i,b_i)\}_{i=1}^m$, $1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i$:

$$\min \sum_{i=1}^m w_i\left[(x-a_i)^2 + (y-b_i)^2\right] = \min \sum_{i=1}^m w_i(x-a_i)^2 + \min \sum_{i=1}^m w_i(y-b_i)^2$$

$$\Rightarrow \frac{d}{dx} \sum_{i=1}^m w_i(x-a_i)^2 = 0 = \cancel{2}\sum_{i=1}^m w_i(x-a_i) \Rightarrow x = \sum w_i a_i / \sum w_i$$

$$\frac{d}{dy} \sum_{i=1}^m w_i(y-b_i)^2 = 0 = \cancel{2}\sum_{i=1}^m w_i(y-b_i) \Rightarrow y = \sum w_i b_i / \sum w_i.$$

**2.2.5 Thm** (explicit solution formula for $1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i$): Let $V=\{(a_i,b_i)\}_{i=1}^m$

$$\operatorname*{argmin}_{(x,y)\in\mathbb{R}^2} 1/\mathbb{R}^2/\cdot/l_2^2/\Sigma w_i = \left(\sum_{i=1}^m w_i a_i, \sum_{i=1}^m w_i b_i\right) / \sum_{i=1}^m w_i. \quad \square$$

**2.2.5 Ex.** ($l_2$): $V=\{(a_i,b_i)\}_{i=1}^m = \{v_i\}_{i=1}^m$, $1/\mathbb{R}^2/\cdot/l_2/\Sigma$

$$\min_{p=(x,y)\in\mathbb{R}^2} f(p) = \min_{p=(x,y)\in\mathbb{R}^2} \sum_{i=1}^m \sqrt{(x-a_i)^2+(y-b_i)^2} = \min_{p\in\mathbb{R}^2} \sum_{i=1}^m \|p-v_i\|_2$$

- $f$ is convex (as a sum of convex functions)
- $f$ is differentiable at $p \notin V$ and

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(p) = \left(\sum \frac{\cancel{2}(x-a_i)}{\cancel{2}\|p-v_i\|_2}, \sum \frac{y-b_i}{\|p-v_i\|_2}\right) = \sum_{i=1}^m \frac{p-v_i}{\|p-v_i\|_2}.$$

• $\nabla f(p) = \sum_{i=1}^m \frac{p-v_i}{\|p-v_i\|_2} = 0 \Rightarrow \sum \frac{p}{\|p-v_i\|_2} = p \underbrace{\sum_{i=1}^m \frac{1}{\|p-v_i\|_2}}_{=:\lambda(p)} = \sum \frac{v_i}{\|p-v_i\|_2}$

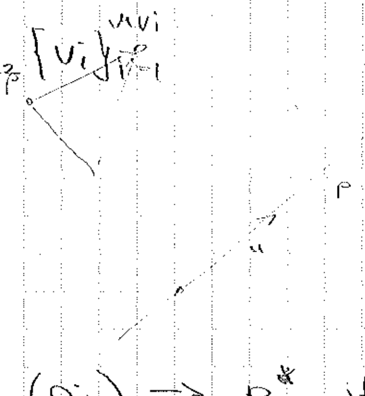
$\Rightarrow p = \frac{1}{\lambda(p)} \sum_{i=1}^m \frac{v_i}{\|p-v_i\|_2} =: T(p)$

• $p_1,\ p_2 := T(p_1),\ p_3 := T(p_2),\ \dots \to p^* = \arg\min_{p\in\mathbb{R}^2} \sum_{i=1}^m \|p-v_i\|_2$

Thm 22.6 (Weiszfeld [1937]): Let $V=\{v_i\}_{i=1}^m \subseteq \mathbb{R}^2$ be a set of points s.t. $\dim V > 1$ (i.e. the points $v_i$ do not lie on a line). Then there exists a unique point

$$p^* = \arg\min_{p\in\mathbb{R}^2} \sum_{i=1}^m \|p-v_i\|_2 =: \mathrm{med}_{\|\cdot\|_2}\{v_i\}_{i=1}^m$$

and $p^*$ is characterized as follows:



a) $p^*\notin V \iff \sum_{i=1}^m \frac{p^*-v_i}{\|p^*-v_i\|_2} = 0$

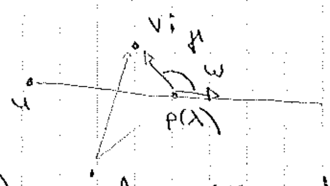
b) $p^* = v_k \iff \left\| \sum_{i=1, i\neq k}^m \frac{p^*-v_i}{\|p^*-v_i\|_2} \right\|_2 \le 1.$

( $p_1\notin V,\ p_{i+1} = T(p_i),\ i=1,2,\dots \Rightarrow (p_i) \to p^*$ if $p_i\notin V\ \forall i$.

Lem. 22.7: Let $p(\lambda) = u+\lambda w$, $\lambda\in\mathbb{R}$, be a line in $\mathbb{R}^2$, $p(\mathbb{R}) \not\supseteq V$. Then

$$g:\mathbb{R}\to\mathbb{R},\ \lambda\mapsto w^T \sum_{i=1}^m \frac{v_i-p(\lambda)}{\|v_i-p(\lambda)\|_2}$$

is strictly monotonously decreasing.




Proof: $w^T \frac{[v_i-p(\lambda)]}{\|v_i-p(\lambda)\|_2} = \|w\|_2 \cos\sphericalangle(w, v_i-p(\lambda))$ decreases and strictly for at least one $v_i$ not on the line. □

Lem. 22.8: There is at most one point satisfying 22.6 a).

Proof: Suppose $p_1, p_2 \in \mathbb{R}^2$, $p_1\neq p_2$ do and consider $p(\lambda) = p_1+\lambda(p_2-p_1)$:

$$\sum_{i=1}^m \frac{\overbrace{p_1}^{p_1+0(p_2-p_1)}-v_i}{\|p_1-v_i\|_2} = \sum_{i=1}^m \frac{\overbrace{p_2}^{=p_1+1(p_2-p_1)}-v_i}{\|p_2-v_i\|_2} = 0$$

$$\Rightarrow (p_2-p_1)^T \sum_{i=1}^m \frac{p_1-v_i}{\|p_1-v_i\|_2} = (p_2-p_1)^T \sum_{i=1}^m \frac{p_2-v_i}{\|p_2-v_i\|_2} \quad ↯. \ □$$

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Lem. 22.9: If $p^*$ satisfies 22.6 a), there is no $v_i$ satisfying 22.6 b).

Proof: Suppose $v_1$ does and consider $p(\lambda) = v_1 + \lambda \frac{p^*-v_1}{\|p^*-v_1\|_2} \not\supseteq \{v_2, \dots, v_m\}$.

$\Rightarrow \lambda \mapsto \frac{(p^*-v_1)^T}{\|p^*-v_1\|_2} \sum_{i=2}^m \frac{v_i-p(\lambda)}{\|v_i-p(\lambda)\|_2}$ is strictly mon. decreasing

$$\Rightarrow \underbrace{\frac{(p^*-v_1)^T}{\|p^*-v_1\|_2}}_{\|\ \|_2=1} \underbrace{\sum_{i=2}^m \frac{v_i-v_1}{\|v_i-v_1\|_2}}_{\substack{\|\ \|\le 1 \\ \text{by b)}}} > \underbrace{\underbrace{\frac{(p^*-v_1)^T}{\|p^*-v_1\|_2}}_{1} \underbrace{\left[\sum_{i=2}^m \frac{v_i-p^*}{\|v_i-p^*\|_2} + \frac{v_1-p^*}{\|v_1-p^*\|_2}\right]}_{=0} \underbrace{- (p^*-v_1)^T \frac{v_1-p^*}{\|v_1-p^*\|_2^2}}_{=+1}}_{=1} \quad ↯$$

□

**Lem. 2.2.10**: At most one $v_i$ satisfies 2.2.6 b).

**Proof**: Suppose $v_1, v_2$ do and consider $p(\lambda) = v_1 + \lambda \frac{v_2-v_1}{\|v_2-v_1\|_2}$. Then $p(\lambda) \not\ni \{v_3, \dots, v_m\}$ and $\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i-v_1}{\|v_i-v_1\|_2} > \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i-v_2}{\|v_i-v_2\|_2}$

$$\Rightarrow 1 \ge \left\| \sum_{i=2}^m \frac{v_i-v_1}{\|v_i-v_1\|_2} \right\|_2 \ge \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=2}^m \frac{v_i-v_1}{\|v_i-v_1\|_2} = \underbrace{\frac{\|v_2-v_1\|_2^2}{\|v_2-v_1\|_2^2}}_{=+1} + \underbrace{\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i-v_1}{\|v_i-v_1\|_2}}_{\le 0}$$

$$\Rightarrow \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \sum_{i=3}^m \frac{v_i-v_2}{\|v_i-v_2\|_2} < 0$$

$$\Rightarrow \underbrace{\underbrace{\frac{(v_2-v_1)^T}{\|v_2-v_1\|_2} \frac{v_1-v_2}{\|v_1-v_2\|_2}}_{=-1} + \frac{(v_2-v_1)^T}{\|v_2-v_1\|_2}\sum_{i=3}^m \frac{v_i-v_2}{\|v_i-v_2\|_2}}_{<0} \Rightarrow \left\| \sum_{\substack{i=1 \\ i\neq 2}}^m \frac{v_i-v_2}{\|v_i-v_2\|_2} \right\|_2 > 1 \ ↯. \ \square$$

**Lem. 2.2.11**: $\sum_{i=1}^m \|p_{j+1}-v_i\|_2 \le \sum_{i=1}^m \|p_j - v_i\|_2$, $j = 1, 2, \dots$, and "=" $\Leftrightarrow p_{j+1} = p_j$.

**Proof**: Let $w_i := \frac{1}{\|p_j - v_i\|_2}$, $i = 1, \dots, m$

$$\overset{\text{Thm. 2.2.5}}{\Rightarrow} \operatorname{argmin}_{p\in\mathbb{R}^2} \sum_{i=1}^m w_i \|p - v_i\|_2^2 = \frac{1}{\underbrace{\sum_{i=1}^m w_i}_{=\lambda(p_j)}} \sum_{i=1}^m w_i v_i = \frac{1}{\lambda(p_j)} \sum_{i=1}^m \frac{1}{\|p_j - v_i\|_2} v_i$$

$$= T(p_j) = p_{j+1}$$

$$\Rightarrow \sum_{i=1}^m \|p_j - v_i\|_2 = \sum_{i=1}^m \underbrace{w_i}_{=\frac{1}{\|p_j-v_i\|_2}} \|p_j - v_i\|_2^2 \ge \sum_{i=1}^m \frac{1}{\|p_j - v_i\|_2} \|p_{j+1} - v_i\|_2^2$$

$$\|p_{j+1} - v_i\|_2^2 = \left[\|p_j - v_i\|_2 + (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)\right]^2$$

$$= \|p_j - v_i\|_2^2 + 2\|p_j - v_i\|_2 (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2) + (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2$$

$$= \cancel{\sum \|p_j - v_i\|_2} + 2\sum \|p_{j+1} - v_i\|_2 - 2\sum \|p_j - v_i\|_2 + \sum \frac{(\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2}{\|p_j - v_i\|_2}$$

$$\Rightarrow 2\sum \|p_j - v_i\|_2 \ge 2\sum \|p_{j+1} - v_i\|_2 + \underbrace{\sum (\|p_{j+1} - v_i\|_2 - \|p_j - v_i\|_2)^2 / \|p_j - v_i\|_2}_{\ge 0} \quad \square$$

**Lem. 2.2.12**: $(p_i)$ is bounded and hence has accumulation points.

**Proof**: $p_i \in \operatorname{conv} V$, $i = 2, 3, \dots$ (as centers of gravity). $\square$

**Lem. 2.2.13**: $T(p) = p$ for any accumulation point $p \notin V$ of $(p_i)$.

**Proof**: Suppose $T(p) = p'$, then $p'$ is also an accumulation point of $(p_i)$ as $T$ is continuous (images of points near $p$ are near $p'$), and $\sum \|p - v_i\| > \sum \|p' - v_i\|$, while $\sum \|p_j - v_i\| \to \sum \|p - v_i\| = \sum \|p' - v_i\|$ ↯ $\square$

Lem. 2.2.14: Any accumulation point $p \notin V$ of $(p_j)$ satisfies 2.2.6 a).

Proof: Let $p = (x, y)$. Then $p = T(p)$

$$\Rightarrow x = \frac{\sum_i \frac{a_i}{\|p - v_i\|_2}}{\sum_i \frac{1}{\|p - v_i\|_2}} \Leftrightarrow \frac{\sum_i \frac{x - a_i}{\|p - v_i\|_2}}{\sum_i \frac{1}{\|p - v_i\|_2}} = 0 \Rightarrow \sum_i \frac{p - v_i}{\|p - v_i\|_2} = 0. \square$$

Cor. 2.2.15: $(p_i)$ admits at most one accumulation point $p \notin V$.

Proof: Lem. 2.2.8 states uniqueness. $\square$

Lem. 2.2.15: If $v_i$ is an accumulation point of $(p_i)$, it is the only condensation point.

W.l.o.g. let $v_i = v_1$.

Proof: Other accumulation points can only be $v_2, \dots, v_m$ and the unique point $p$ satisfying 2.2.6 a) (if it exists) $\Rightarrow \exists\, U_\varepsilon(v_1) \not\ni \{v_2, \dots, v_m, p\}$.

Consider $j_1, j_2, \dots$ s.t. $p_{j_k} \in U_\varepsilon(v_1)$, $p_{j_{k+1}} \notin U_\varepsilon(v_1)$, $k = 1, 2, \dots$

$$\Rightarrow \|p_{j_k} - v_1\|_2 \to 0, \quad \|p_{j_k} - v_i\|_2 \to \|v_1 - v_i\|_2, \quad i = 2, \dots, m \qquad (1)$$

$$\Rightarrow \frac{\|p_{j_{k+1}} - v_1\|_2}{\|p_{j_k} - v_1\|_2} > \frac{\varepsilon}{\|p_{j_k} - v_1\|_2} \to \infty$$

W.l.o.g. let $v_1 = 0$. Then

$$\frac{\|p_{j_{k+1}} - v_1\|_2}{\|p_{j_k} - v_1\|_2} = \frac{\|T(p_{j_k})\|_2}{\|p_{j_k}\|_2} = \frac{\left\| \sum_{i=2}^m \frac{v_i}{\|p_{j_k} - v_i\|_2} \right\|_2}{\underbrace{\sum_{i=1}^m \frac{1}{\|p_{j_k} - v_i\|_2}}_{= \frac{1}{\|p_{j_k}\|_2} + \sum_{i=2}^m \frac{1}{\|p_{j_k} - v_i\|_2}}} \cdot \frac{1}{\|p_{j_k}\|_2}$$

$$= \frac{\left\| \sum_{i=2}^m \frac{v_i}{\|p_{j_k} - v_i\|_2} \right\|_2 \to \|v_i\|_2}{1 + \sum_{i=2}^m \frac{\|p_{j_k}\|_2 \to 0}{\|p_{j_k} - v_i\|_2 \to \|v_1 - v_i\|_2 = \|v_i\|_2}}$$

$$\to \left\| \sum_{i=2}^m \frac{v_i}{\|v_i\|_2} \right\|_2 =: \nu < \infty \ \lightning. \ \square \qquad (2)$$

Lem. 2.2.16: If $v_i$ is an accumulation point of $(p_i)$, it satisfies 2.2.6 b).

W.l.o.g. let $v_i = v_1 = 0$. By Lem. 2.2.15, $v_1$ is the only accumulation point of $(p_i)$

$$\Rightarrow p_j \to v_1 \Rightarrow \|p_j - v_1\|_2 \to 0$$

$$(1) \Rightarrow \|p_j - v_i\|_2 \to \|v_1 - v_2\|_2 \qquad p_j \to v_1$$

$$(2) \Rightarrow \frac{\|p_{j+1} - v_1\|_2}{\|p_j - v_1\|_2} \to \underbrace{\left\| \sum_{i=2}^m \frac{v_i}{\|v_i\|_2} \right\|_2}_{= \left\| \sum_{i=2}^m \frac{v_1 - v_i}{\|v_1 - v_i\|_2} \right\|_2} = \nu < 1 \quad (2.2.6\,b)) \ \square$$

Lem. 22.17: Suppose $p_j \notin V$, $j=1,2,\ldots$ Then $(p_j)$ converges to a point $p^*$ satisfying 22.6 a) or b). Moreover, $p^*$ does not depend on $p_1$.

Proof: $(p_j)$ has an accumulation point $p^*$ by Lem. 22.12. If $p^* \in V$, it is unique by Lem 22.16 and satisfies 22.6 b) by Lem. 22.13. If $p^* \notin V$, it is unique by Lem. 22.15 and satisfies 22.6 a) by Lem. 22.14.

Let $p_1' \neq p_1$ and suppose $\lim p_j' = p' \neq p^*$. We have one of the following relations:

a) $\displaystyle\sum_{i=1}^{m} \frac{p^*-v_i}{\|p^*-v_i\|_2} = 0$, $\displaystyle\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} = 0 \overset{\text{Lem. 22.8}}{\Rightarrow} ⚡$

b) " , $\displaystyle\left\|\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2}\right\|_2 \leq 1 \overset{\text{Lem. 22.9}}{\Rightarrow} ⚡$

c) $\displaystyle\left\|\sum_{i=1}^{m} \frac{p^*-v_i}{\|p^*-v_i\|_2}\right\|_2 \leq 1$, $\displaystyle\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2} = 0 \overset{\text{Lem. 22.9}}{\Rightarrow} ⚡$

d) " , $\displaystyle\left\|\sum_{i=1}^{m} \frac{p'-v_i}{\|p'-v_i\|_2}\right\|_2 \leq 1 \overset{\text{Lem. 22.10}}{\Rightarrow} ⚡$ $\square$

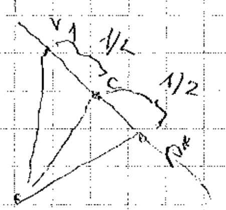
Prop. 22.18: $\lim p_j = p^* = \operatorname{argmin} \displaystyle\sum_{i=1}^{m} \|p-v_i\|_2$ if $p_j \notin V$, $j=1,2,\ldots$

Proof: a) For $x \neq p^*$, $x \notin V$, let $x_1 = x$, $x_2 = T(x_1)$, ...

$$\sum \|\underbrace{x}_{=x_1}-v_i\|_2 \overset{\text{Lem. 22.11}}{>} \sum \|x_2-v_i\|_2 > \ldots = \lim \sum \|x_j - v_i\|_2 = \sum \|p-v_i\|_2 .$$

b) For $x = v_i \neq p^*$, w.l.o.g. $x = v_1 \neq p^*$, suppose $\sum \|\underbrace{x}_{=v_1} - v_i\|_2 \leq \sum \|p^* - v_i\|_2$ (3)

Let $c$ be the median of $v_1$ and $p^*$ and consider the $\triangle\, v_1, p^*, v_i$, $i = 2,\ldots,m$. In any $\triangle$




$\|v_i - c\|_2 \leq (\|v_i - v_1\|_2 + \|v_i - p^*\|_2)/2$ and "=" $\Leftrightarrow v_i \in \operatorname{lin}\{v_1, p^*\}$

$$\Rightarrow \sum \|c - v_i\|_2 \leq \frac{\sum \|v_1 - v_i\|_2 + \sum \|p^* - v_i\|_2}{2} \leq \sum \|p^* - v_i\|_2$$

If $c \notin V \overset{a)}{\Rightarrow} ⚡$. If $c \in V$ consider $c' = \operatorname{med}\{c, p^*\}$ and so on until $c' \notin V$. $\square$

Rem. 22.19: a) $p_j \in V$ can be repaired (e.g. by choosing $p_1$ appropriately.

b) There are solution formulas for $m = 3, 4$. $m = 5$ cannot be solved by a formula involving only elementary algebraic operations $(+/-, \times/\div, \sqrt{\ })$.

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